VII. *Problems concerning Interpolations.* *By* Edward
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1779.
M R. BRIGGS was the firſt perſon, I believe,
that invented a method of differences
for interpolating logarithms at ſmall intervals from each
other: his principles were followed by REGINALD and
MOVTON in France. Sir ISAAC NEWTON, from the ſame
principles, diſcovered a general and elegant ſolution of
the abovementioned problem: perhaps a ſtill more ele-
gant one on ſome accounts has been ſince diſcovered by
Meſſ. NICHOLE and STIRLING. In the following theorems
the ſame problem is reſolved and rendered ſomewhat
more general, without having any recourſe to finding the
ſucceſſive differences.

THEOREM I.

Aſſume an equation $a+bx+cx^2+dx^3 \ldots\ldots x^{n-1}=y$,
in which the co-efficients $a, b, c, d, e,$ &c. are invariable;

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let

let α, β, γ, δ, ε, &c. denote n values of the unknown quantity x, whose correspondent values of y let be represented by s^α, s^β, s^γ, s^δ, s^ε, &c. Then will the equation $a + bx + c.x^2 + d.x^3 + ex^4 \ldots x^{n-1} = y =$

$$\frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.} \times s^\alpha + \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\varepsilon} \times \&c.} \times s^\beta$$

$$+ \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\varepsilon} \times \&c.} \times s^\gamma + \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\varepsilon} \times \&c.}{\overline{\delta-\alpha} \times \overline{\delta-\beta} \times \overline{\delta-\gamma} \times \overline{\delta-\varepsilon} \times \&c.} \times s^\delta$$

$$+ \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \&c.}{\overline{\varepsilon-\alpha} \times \overline{\varepsilon-\beta} \times \overline{\varepsilon-\gamma} \times \overline{\varepsilon-\delta} \times \&c.} \times s^\varepsilon + \&c.$$

DEMONSTRATION.

Write α for x in the equation $y =$

$$\frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.} \times s^\alpha + \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\varepsilon} \times \&c.} \times s^\beta +$$

&c.; and all the terms but the first in the resulting equation will vanish, for each of them contains in its numerator a factor $x-\alpha = \alpha-\alpha = o$; and the equation will become $y = \dfrac{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.} \times s^\alpha = s^\alpha$. In the same manner, by writing β, γ, δ, ε, &c. successively for x in the given equation it may be proved, that when x is equal to β, γ, δ, ε, &c. then will y become respectively s^β, s^γ, s^δ, s^ε, which was to be demonstrated.

2. Assume $y = ax^r + bx^{r+s} + cx^{r+2s} + dx^{r+3s} \ldots x^{r+\overline{n-1}s}$; and when x becomes α, β, γ, δ, ε, &c. let y become respectively

fpe&ively s^{α}, s^{β}, s^{γ}, s^{δ}, s^{ϵ}, &c.; then will $y =$

$$\frac{\overline{x^{r} \times x^{s} - \beta^{s}} \times \overline{x^{s} - \gamma^{s}} \times \overline{x^{s} - \delta^{s}} \times \overline{x^{s} - \epsilon^{s}} \times \&c.}{\overline{\alpha^{r} \times \alpha^{s} - \beta^{s}} \times \overline{\alpha^{s} - \gamma^{s}} \times \overline{\alpha^{s} - \delta^{s}} \times \overline{\alpha^{s} - \epsilon^{s}} \times \&c.} \times s^{\alpha}$$

$$+ \frac{\overline{x^{r} \times x^{s} - \alpha^{s}} \times \overline{x^{s} - \gamma^{s}} \times \overline{x^{s} - \delta^{s}} \times \overline{x^{s} - \epsilon^{s}} \times \&c.}{\overline{\beta^{r} \times \beta^{s} - \alpha^{s}} \times \overline{\beta^{s} - \gamma^{s}} \times \overline{\beta^{s} - \delta^{s}} \times \overline{\beta^{s} - \epsilon^{s}} \times \&c.} \times s^{\beta}$$

$$+ \frac{\overline{x^{r} \times x^{s} - \alpha^{s}} \times \overline{x^{s} - \beta^{s}} \times \overline{x^{s} - \delta^{s}} \times \overline{x^{s} - \epsilon^{s}} \times \&c.}{\overline{\gamma^{r} \times \gamma^{s} - \alpha^{s}} \times \overline{\gamma^{s} - \beta^{s}} \times \overline{\gamma^{s} - \delta^{s}} \times \overline{\gamma^{s} - \epsilon^{s}} \times \&c.} \times s^{\gamma} + \&c.$$

This may be demonftrated in the fame manner as the preceding theorem, by writing α, β, γ, δ, ϵ, &c. fuc-ceffively for x.

PROBLEM.

Let there be n values α, β, γ, δ, ϵ, &c. of the quantity x,, to which the n values s^{α}, s^{β}, s^{γ}, s^{δ}, s^{ϵ}, &c. of the quan- tity y correfpond; fuppofe thefe quantities to be found by any fun&ion X of the quantity x; let π, ρ, σ, τ, &c. be values of the quantities x, to which s^{π}, s^{ϵ}, s^{σ}, s^{τ}, &c. values of the quantity y correfpond: for x fubftitute its abovementioned values π, ρ, σ, τ, &c. in the fun&ion X, and let the quantities refulting be s^{π}, s^{ϵ}, s^{σ}, s^{τ}, &c. not equal to the preceding s^{π}, s^{ϵ}, s^{σ}, s^{τ}, &c. refpe&ively; to find a quantity which added to the fun&ion X fhall not only give the true values of the quantity y correfponding to the values α, β, γ, δ, ϵ, &c. of the quantity x, but alfo

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corresponding to the values π, ϱ, σ, τ, &c. of the above-mentioned quantity x.

Assume $s^\pi - s^\pi = T^\pi$, $s^\varrho - s^\varrho = T^\varrho$, $s^\sigma - s^\sigma = T^\sigma$, $s^\tau - s^\tau = T^\tau$, &c.; then the errors of the function X will be respectively T^π, T^ϱ, T^σ, T^τ, &c.; and the correcting quantity sought may be

$$\frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\iota} \times \&c.}{\overline{\pi-\alpha} \times \overline{\pi-\beta} \times \overline{\pi-\gamma} \times \overline{\pi-\delta} \times \overline{\pi-\iota} \times \&c.} \times \frac{\overline{x-\varrho} \times \overline{x-\sigma} \times \overline{x-\tau} \times \&c.}{\overline{\pi-\varrho} \times \overline{\pi-\sigma} \times \overline{\pi-\tau} \times \&c.} \times T^\pi$$

$$+\frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\iota} \times \&c.}{\overline{\varrho-\alpha} \times \overline{\varrho-\beta} \times \overline{\varrho-\gamma} \times \overline{\varrho-\delta} \times \overline{\varrho-\iota} \times \&c.} \times \frac{\overline{x-\pi} \times \overline{x-\sigma} \times \overline{x-\tau} \times \&c.}{\overline{\varrho-\pi} \times \overline{\varrho-\sigma} \times \overline{\varrho-\tau} \times \&c.} \times T^\varrho$$

$$+\frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\iota} \times \&c.}{\overline{\sigma-\alpha} \times \overline{\sigma-\beta} \times \overline{\sigma-\gamma} \times \overline{\sigma-\delta} \times \overline{\sigma-\iota} \times \&c.} \times \frac{\overline{x-\pi} \times \overline{x-\varrho} \times \overline{x-\tau} \times \&c.}{\overline{\sigma-\pi} \times \overline{\sigma-\varrho} \times \overline{\sigma-\tau} \times \&c.} \times T^\sigma$$

$$+\frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\iota} \times \&c.}{\overline{\tau-\alpha} \times \overline{\tau-\beta} \times \overline{\tau-\gamma} \times \overline{\tau-\delta} \times \overline{\tau-\iota} \times \&c.} \times \frac{\overline{x-\pi} \times \overline{x-\varrho} \times \overline{x-\sigma} \times \&c.}{\overline{\tau-\pi} \times \overline{\ \ -\ } \times \overline{\tau-\sigma} \times \&c.} \times T^\tau$$

$+$ &c.

Aliter.

Let $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times$ &c. $\times \overline{x-\pi}$ $\times \overline{x-\varrho} \times \overline{x-\sigma} \times \overline{x-\tau} \times$ &c. $= N$; $\overline{\pi-\alpha} \times \overline{\pi-\beta} \times \overline{\pi-\gamma} \times \overline{\pi-\delta} \times \overline{\pi-\varepsilon}$ $\times$ &c. $\times \overline{\pi-\varrho} \times \overline{\pi-\sigma} \times \overline{\pi-\tau} \times$ &c. $= \Pi$; $\overline{\varrho-\alpha} \times \overline{\varrho-\beta} \times \overline{\varrho-\gamma} \times \overline{\varrho-\delta} \times$ $\overline{\varrho-\varepsilon} \times$ &c. $\times \overline{\varrho-\pi} \times \overline{\varrho-\sigma} \times \overline{\varrho-\tau} \times$ &c. $= P$; $\overline{\sigma-\alpha} \times \overline{\sigma-\beta} \times \overline{\sigma-\gamma} \times$ $\overline{\sigma-\delta} \times \overline{\sigma-\varepsilon} \times$ &c. $\times \overline{\sigma-\pi} \times \overline{\sigma-\varrho} \times \overline{\sigma-\tau} \times$ &c. $= \Sigma$; $\overline{\tau-\alpha} \times \overline{\tau-\beta} \times$ $\overline{\tau-\gamma} \times \overline{\tau-\delta} \times \overline{\tau-\varepsilon} \times$ &c. $\times \overline{\tau-\pi} \times \overline{\tau-\varrho} \times \overline{\tau-\sigma} \times$ &c. $= T$, &c.; then may the correcting quantity sought be $N \left(\dfrac{T^\tau}{\Pi \times \overline{x-\pi}} + \dfrac{T^\varrho}{P \times \overline{x-\varrho}} \right.$

$\left. + \dfrac{T^\sigma}{\Sigma \times \overline{x-\sigma}} + \dfrac{T^\tau}{T \times \overline{x-\tau}} + \&c. \right).$

This

This problem may be demonſtrated in the ſame man-
ner as the preceding theorems, by writing for x in the
correcting quantity ſucceſſively its values π, ϱ, σ, τ, &c.

2. For the correcting quantity ſought may be aſſumed

the quantity $\dfrac{\overline{x^s - \alpha^s} \times \overline{x^s - \beta^s} \times \overline{x^s - \gamma^s} \times \overline{x^s - \delta^s} \times \&c. \times \overline{x^r \times x^s - \varrho^s} \times \overline{x^s - \sigma^s}}{\overline{\pi^s - \alpha^s} \times \overline{\pi^s - \beta^s} \times \overline{\pi^s - \gamma^s} \times \overline{\pi^s - \delta^s} \times \&c. \times \overline{\pi^r \times \pi^s - \varrho^s} \times \overline{\pi^s - \sigma^s}}$

$\dfrac{\times \overline{x^s - \tau^s} \times \&c.}{\times \overline{\pi^s - \tau^s} \times \&c.} \times \mathrm{T}^\pi + \dfrac{\overline{x^s - \alpha^s} \times \overline{x^s - \beta^s} \times \overline{x^s - \gamma^s} \times \overline{x^s - \delta^s} \times \&c. \times \overline{x^r \times x^s - \pi^s}}{\overline{\varrho^s - \alpha^s} \times \overline{\varrho^s - \beta^s} \times \overline{\varrho^s - \gamma^s} \times \overline{\varrho^s - \delta^s} \times \&c. \times \overline{\varrho^r \times \varrho^s - \pi^s}}$

$\dfrac{\times \overline{x^s - \sigma^s} \times \overline{x^s - \tau^s} \times \&c.}{\times \overline{\varrho^s - \sigma^s} \times \overline{\varrho^s - \tau^s} \times \&c.} \times \mathrm{T}^\varrho + \&c.$

3. In general, let z be any quantity which is $= o$,
when x becomes either α, β, γ, δ, ε, &c.: let z become
ſucceſſively A, B, C, D, &c. when x becomes π, ϱ, σ, τ,
&c. reſpectively. When x either $= \varrho$, σ, τ, &c. let $\Pi = o$;
but if $x = \pi$, let $\Pi = p$: in the ſame manner when x either
$= \pi$, σ, τ, &c. let P$=o$; but when $x = \varrho$ let P$=r$: and ſimi-
larly, let $\Sigma = o$ when x is either π, ϱ, τ, &c.; but when
$x = \sigma$ let $\Sigma = s$: and likewiſe, when x is either π, ϱ, σ, &c.
let T$=o$; but when $x = \tau$ let T$=t$: &c. then for the cor-
recting quantity ſought may be aſſumed $\dfrac{z}{A} \times \dfrac{\Pi}{p} \times \mathrm{T}^\pi +$

$\dfrac{z}{B} \times \dfrac{P}{r} \times \mathrm{T}^\varrho + \dfrac{z}{C} \times \dfrac{\Sigma}{s} \times \mathrm{T}^\sigma + \dfrac{z}{D} \times \dfrac{T}{t} \times \mathrm{T}^\tau + \&c.$

THEOREM.

Affume (n) quantities α, β, γ, δ, ε, &c. then will the fum of all the (n) quantities of the following kind

$$\frac{\alpha^m}{\overline{a-\beta}\times\overline{a-\gamma}\times\overline{a-\delta}\times\overline{a-\iota}\times \text{ \&c.}} + \frac{\beta^m}{\overline{\beta-a}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\iota}\times \text{ \&c.}}$$

$$+ \frac{\gamma^m}{\overline{\gamma-a}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\iota}\times \text{ \&c.}} + \frac{\delta^m}{\overline{\delta-a}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\iota}\times \text{ \&c.}}$$

$$+ \frac{\iota^m}{\overline{\iota-a}\times\overline{\iota-\beta}\times\overline{\iota-\gamma}\times\overline{\iota-\delta}\times \text{ \&c.}} + \text{\&c.} = 0,$$ if m be any whole number lefs than $n-1$; but if $m=n-1$, then will the above mentioned fum $=1$. In general, the fum of the n terms

$$\frac{\alpha^m(\beta\gamma\delta\text{\&c.}+\beta\gamma\iota\text{\&c.}+\beta\delta\iota+\gamma\delta\iota\text{\&c.}+\text{\&c.})}{\overline{a-\beta}\times\overline{a-\gamma}\times\overline{a-\delta}\times\overline{a-\iota}\times \text{ \&c.}} +$$

$$\frac{\beta^m(a\gamma\delta\text{\&c.}+a\gamma\iota\text{\&c.}+a\delta\iota\text{\&c.}+\gamma\delta\iota\text{\&c.}+\text{\&c.})}{\overline{\beta-a}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\iota}\times \text{ \&c.}} +$$

$$\frac{\gamma^m(a\beta\delta\text{\&c.}+a\beta\iota\text{\&c.}+a\delta\iota\text{\&c.}+\beta\delta\iota\text{\&c.}+\text{\&c.})}{\overline{\gamma-a}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\iota}\text{\&c.}} +$$

$$\frac{\delta^m(a\beta\gamma\text{\&c.}+a\beta\iota\text{\&c.}+a\gamma\iota\text{\&c.}+\text{\&c.})}{\overline{\delta-a}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\iota}\times \text{ \&c.}} +$$

$$\frac{\iota^m(a\beta\gamma\text{\&c.}+a\beta\delta\text{\&c.}+a\gamma\delta\text{\&c.}+\beta\gamma\delta\text{\&c.}+\text{\&c.})}{\overline{\iota-a}\times\overline{\iota-\beta}\times\overline{\iota-\gamma}\times\overline{\iota-\delta}\text{\&c.}} + \text{\&c.} = 0,$$

if m be lefs than n, and $m+r$ not equal to $n-1$, where r is equal to the number of letters contained in each of the contents above mentioned $\beta\gamma\delta$, &c. $\beta\gamma\varepsilon$, &c. $\beta\delta\varepsilon$, &c. $\gamma\delta\varepsilon$, &c. &c. &c. refpectively: but if $m+r=n-1$, then will the above mentioned fum $=\pm 1$; it will be $+1$ if r be an even number, otherwife -1.

DEMONSTRATION.

Suppofe $a+b\alpha+c\alpha^2+d\alpha^3+e\alpha^4+$ &c. $=s^\alpha$,

$\quad a+b\beta+c\beta^2+d\beta^3+e\beta^4+$ &c. $=s^\beta$,

$\quad a+b\gamma+c\gamma^2+d\gamma^3+e\gamma^4+$ &c. $=s^\gamma$,

$\quad a+b\delta+c\delta^2+d\delta^3+e\delta^4+$ &c. $=s^\delta$,

$\quad a+b\varepsilon+c\varepsilon^2+d\varepsilon^3+e\varepsilon^4+$ &c. $=s^\varepsilon$, multiply thefe equations into A, B, C, D, E, &c. unknown co-effi-cients to be inveftigated, and there refult

$A\times s^\alpha = Aa+Ab\alpha+Ac\alpha^2+Ad\alpha^3+Ae\alpha^4+$&c.

$B\times s^\beta = Ba+Bb\beta+Bc\beta^2+Bd\beta^3+Be\beta^4+$&c.

$C\times s^\gamma = Ca+Cb\gamma+Cc\gamma^2+Cd\gamma^3+Cd\gamma^4+$&c.

$D\times s^\delta = Da+Db\delta+Dc\delta^2+Dd\delta^3+Dd\delta^4+$&c.

$E\times s^\varepsilon = Ea+Eb\varepsilon+Ec\varepsilon^2+Ed\varepsilon^3+Ee\varepsilon^4+$ &c. &c. &c.

Now fuppofe $As^\alpha+Bs^\beta+Cs^\gamma+Ds^\delta+Es^\varepsilon+$ &c. $=a+bx+cx^2+dx^3+ex^4+$&c. and the correfpondent parts refpectively equal to each other; that is, $a(A+B+C+D+E+$ &c.$)=a$; $b(A\alpha+B\beta+C\gamma+D\delta+E\varepsilon+$ &c.$)=bx$; $A\alpha^2+B\beta^2+C\gamma^2+D\delta^2+E\varepsilon^2+$ &c. $=x^2$; $A\alpha^3+B\beta^3+C\gamma^3+D\delta^3+E\varepsilon^3+$ &c. $=x^3$; $A\alpha^4+B\beta^4+C\gamma^4+D\delta^4+E\varepsilon^4+$ &c. $=x^4$, &c.: But it follows from Theorem I. that (if $As^\alpha+Bs^\beta+Cs^\gamma+Ds^\delta+Es^\varepsilon+$&c.

$=a+bx+cx^2+dx^3+ex^4+$&c.$)A=\dfrac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\ \&c.}{\overline{a-\beta}\times\overline{a-\gamma}\times\overline{a-\delta}\times\overline{a-\varepsilon}\ \&c.}$,''

$$B = \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \times \&c.}, \quad C = \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \times \&c.},$$

$$D = \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\epsilon} \times \&c.}{\overline{\delta-\alpha} \times \overline{\delta-\beta} \times \overline{\delta-\gamma} \times \overline{\delta-\epsilon} \times \&c.}, \quad E = \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \&c.}{\overline{\epsilon-\alpha} \times \overline{\epsilon-\beta} \times \overline{\epsilon-\gamma} \times \overline{\epsilon-\delta} \times \&c.},$$

&c.: fubftitute thefe values for A, B, C, D, E, &c. refpec-
tively in the preceding equations (A + B + C + D + E + &c. = 1,

Aα + Bβ + Cγ + Dδ + Eϵ + &c. = x,   Aα^2 + Bβ^2 + Cγ^2 + Dδ^2 +

Eϵ^2 + &c. = x^2,   Aα^3 + Bβ^3 + Cγ^3 + Dδ^3 + Eϵ^3 + &c. = x^3, &c.)

and there refult the equations (1) $\dfrac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \&c.}$

$$+ \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \&c.} + \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \&c.} + \&c. = 1;$$

$$(2)\, \alpha \times \frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \times \&c.} + \beta \times \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \times \&c.}$$

$$+ \gamma \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \times \&c.} + \&c. = x;$$

$$(3)\, \alpha^2 \times \frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \&c.} + \beta^2 \times \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \&c.}$$

$$+ \gamma^2 \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \&c.} = x^2; \quad \text{and in general,}$$

$$\alpha^m \times \frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \&c.} + \beta^m \times \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \&c.}$$

$$+ \gamma^m \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \&c.} + \delta^m \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\epsilon} \&c.}{\overline{\delta-\alpha} \times \overline{\delta-\beta} \times \overline{\delta-\gamma} \times \overline{\delta-\epsilon} \&c.}$$

+ &c. = x^m, whatever may be the values of the quantities
x; $\alpha, \beta, \gamma, \delta, \epsilon$, &c.: reduce all thefe fractions into terms,
proceeding according to the dimenfions of the quantity
x, and it is evident, that the fum of all the fractions mul-
tiplied

tiplied into any dimenfion of x not equal to m will be $= o$; but the fum of all the fractions multiplied into x^m will be $= 1$: from this propofition the theorem is eafily deduced.

I have invented and demonftrated from different principles to the preceding the firft part of this theorem, a particular cafe of which was publifhed by me many years ago.

From this theorem may eafily be deduced feveral others of a fimilar nature.